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Seventh Memoir on the Partition of Numbers. A Detailed Study of the Enumeration of the Partitions of Multipartite Numbers

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IV. *Seventh Memoir on the Partition of Numbers.*

A Detailed Study of the Enumeration of the Partitions of Multipartite Numbers.

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Introduction.

In this paper I give the complete solution of the problem of the partition of multipartite numbers. This is the same subject as that named by Sylvester, "Compound Denumeration." Twenty-nine years have elapsed since I announced that the algebra of symmetric functions is co-extensive with the grand problems of the combinatory analysis. The theory of symmetric function supplies generating functions which enumerate all the combinations, while the operators of Hammond* are the instruments which are effective in actually evaluating the coefficients of the terms of the expanded generating functions. When these operators fell from the hands of Hammond, they were already of much service as mining tools in extracting the ore from the mine field of symmetric functions; but they were only partially adequate. They required sharpening and general adaptation to the work in hand. The first step was to decompose an operator of given order into the sum of a number of operators in correspondence with every partition of the number which defines the order. Since there is a Hammond operator corresponding to every positive integer, this process resulted in there being an operator in correspondence with every partition of every integer. The outcome of this decomposition was that the operators were able to deal with the symmetric operands in a much more effective manner. The surface material of the mine could not only be removed, but the strata to a considerable depth could be dealt with. But this was not sufficient. It became necessary to effect a further decomposition by showing that every partition operator could be represented by a sum of composition operators. There emerged a composition operator in correspondence with every permutation of the parts of the partition of the operator. The operators at once became effective in dealing with the material in the lower strata of the mine field. The operators had, in fact, been handled with particular reference to the operands with which they were to be associated. It was now necessary to deal with the material of the mine with particular reference to the tools which had

* 'Proc. Lond. Math. Soc.,' 1883, vol. xiv., pp. 119–129.

been forged. To evaluate the coefficients we have to operate repeatedly with the appropriate operators until a numerical result is reached. In order to accomplish this with facility and to establish laws we have to put the generating functions in such a form that these operations are carried out in a regular and simple manner. To make my meaning clear, I will instance the case of the simple operation of differentiation ∂_x and the exponential function e^{ax}. We have

$$\partial_x e^{ax} = ae^{ax},$$

the effect of the operation being, to merely multiply the operand by the numerical magnitude a.

Thence

$$\partial_x{}^n e^{ax} = a^n e^{ax}$$

and we arrive at the conclusion, that if a given operand, a function of x, could be expressed as a linear function of exponential functions of x, the r times repeated operation of ∂_x could take place with facility upon each term of the linear function, and a general law for the repeated operation of ∂_x upon the operand would be obtainable. This reflection suggests the possibility of finding symmetric function operands in a form which will enable the repeated performance of HAMMOND's operators in a practically effective manner. It is quite certain that any such operand must possess at least two properties in common with the exponential function: (i) its first term must be unity; (ii) it must contain an infinite number of terms. The first step was to find a symmetric function $\mathfrak{Q}_1$ of the elements $\alpha, \beta, \gamma, \ldots$ such that the effect of every Hammond operator upon it is to leave it unchanged; or, as I prefer to say, to multiply it by unity. $\mathfrak{Q}_1$ is, in fact, the sum of unity and the whole of the monomial symmetric functions $\Sigma\alpha^p\beta^q\gamma^k \ldots \equiv (pqr \ldots)$. It is in the partition notation

$$\mathfrak{Q}_1 = 1+(1)+(2)+(1^2)+(3)+(21)+(1^3)+\ldots \text{ ad inf.}$$

It was then found that the effect of any Hammond operator upon any power of $\mathfrak{Q}_1$ is merely to multiply it by a positive integer. It then appeared that, denoting $\mathfrak{Q}_1$ by $\mathrm{F}(\alpha, \beta, \gamma, \ldots)$, the function

$$\mathfrak{Q}_i = \mathrm{F}(\alpha^i, \beta^i, \gamma^i, \ldots)$$

possesses properties of a character similar to those appertaining to $\mathfrak{Q}_1$. The fact is that any Hammond operator when performed upon any power of $\mathfrak{Q}_i$, say $\mathfrak{Q}_i^{k_i}$ has the effect of merely multiplying it by an integer, which may exceptionally be zero. Finally the important fact emerged that the performance of any Hammond operator upon the product

$$\mathfrak{Q}_1^{k_1}\mathfrak{Q}_2^{k_2}\ldots\mathfrak{Q}_i^{k_i}\ldots,$$

where $k_1, k_2, \ldots k_i, \ldots$ may, each of them, be zero or any positive integer, is merely to multiply it by a positive integer, which may, exceptionally, be zero

This discovery involves the complete enumerative solution of the unrestricted partition of multipartite numbers into a given number of parts. The reason of this is that the enumerating symmetric function generating function can be expanded in ascending powers of the functions $\mathfrak{Q}_1, \mathfrak{Q}_2, \mathfrak{Q}_3, \ldots$. On every term of this expansion the repeated performance of Hammond operators is practically effective and is successful in forcing out the sought numerical coefficients. When the magnitudes of the integer constituents of the multipartite parts are restricted in any manner there exists similarly an appropriate series of symmetric functions,

$$U_1, U_2, \ldots U_i \ldots,$$

the formation of which is explained in the paper, which in their properties are analogous to the series $\mathfrak{Q}_1, \mathfrak{Q}_2, \ldots \mathfrak{Q}_i, \ldots$. This circumstance involves the complete enumerative solution when the magnitudes of the constituents of the parts are restricted in any manner whatever.

SECTION I.

The Partition of Multipartite Numbers.

Art. 1. One of the problems which has engaged the attention of writers on the combinatory analysis is that of enumerating the different modes of exhibiting a given composite integer as the product of a given number of factors. For instance, the number 30, which is the product of three unrepeated primes, can be given as the product of two factors in the three ways,

$$2 \times 15, \qquad 3 \times 10, \qquad 5 \times 6.$$

When the given composite number is a product of different primes the question is very easy and is completely solved by means of the generating function

$$1/(1-x)(1-2x)\ldots(1-kx).$$

In the ascending expansions the coefficient of x^{q-k} is the number of ways of factorizing a number, which is the product of q different primes, into exactly k factors.*

Generating functions of the same character have also been obtained for some other simple forms of the composite number such as $p_1^2 p_2 \ldots p_q$, $p_1^2 p_2^2 p_3 \ldots p_q$; $p_1, p_2, \ldots$ denoting primes.

It is, of course, obvious that the absolute magnitudes of the prime factors have nothing whatever to do with the question, which necessarily appertains to the exponents of the primes and to nothing else.

* Compare 'NETTO'S Combinatorik,' 1901, pp. 168 *et seq.*

Art. 2. Writers upon the problem have not usually observed that the general question is identical with the partition of a multipartite number into a given number of parts. Thus the problem discussed by NETTO and others is simply the enumeration of the partitions, into exactly k parts, of the multipartite number

$$1111 \ldots q \text{ times repeated.}$$

Ex. gr. when $q = 3$, $k = 2$ (see the two-factor factorization of $2 \times 3 \times 5$ above), we have three partitions of the multipartite 111 into exactly two parts. These partitions are

$$(110, 001), \qquad (101, 010), \qquad (011, 100).$$

In general the enumeration of the factorizations, involving k factors, of the composite number

$$p_1^{m_1} p_2^{m_2} \ldots p_s^{m_s},$$

yields the same number as the enumeration of the partitions of the multipartite number

$$m_1 m_2 \ldots m_s$$

into exactly k (multipartite) parts.

Art. 3. It is the same problem also to enumerate the separations of a given (unipartite) partition. Thus in relation to the partition (321) of the number 6, there is a one-to-one correspondence between the separations which involve two separates and the partitions of the multipartite number 111, which involve exactly two parts.

The separations are in fact

$$(32)(1), \qquad (31)(2), \qquad (21)(3).$$

In general there is a one-to-one correspondence between the separations of the partition

$$(q_1^{m_1} q_2^{m_2} \ldots q_s^{m_s}),$$

which involve k separates, and the partitions of the multipartite number

$$m_1 m_2 \ldots m_s,$$

which involve exactly k parts.

Art. 4. The general question of multipartite partition I have already discussed* by a method of grouping the partitions and a particular theory of distribution. The present investigation which depends upon other principles leads to results of a different and more general character. I showed many years ago† that in regard to the system of infinitely numerous quantities

$$\alpha, \beta, \gamma, \ldots,$$

* 'Phil. Trans. Camb. Phil. Soc.,' vol. xxi., No. xviii., pp. 467–481, 1912.

† 'Proc. Lond. Math. Soc.,' vol. xix., 1887, pp. 220 *et seq.*

the enumerating generating function is the symmetric function

$$\frac{1}{(1-a)} \times \frac{1}{(1-\alpha a)(1-\beta a)(1-\gamma a)\dots}$$
$$\times \frac{1}{(1-\alpha^2 a)(1-\beta^2 a)(1-\gamma^2 a)\dots(1-\alpha\beta a)(1-\alpha\gamma a)(1-\beta\gamma a)\dots}$$
$$\times \frac{1}{(1-\alpha^3 a)\dots(1-\alpha^2\beta a)\dots(1-\alpha\beta\gamma a)\dots}$$
$$\times \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

wherein if h_s denote the sum of homogeneous products of weight s of the quantities $\alpha, \beta, \gamma, \dots$, the $s-1^{\text{th}}$ fractional factor of the generating function possesses a denominator factor corresponding to every separate term of h_s. The function is to be developed in ascending powers of a and, replacing for the moment the series $\alpha, \beta, \gamma, \dots$ by $\alpha_1, \alpha_2, \alpha_3, \dots$, we seek the coefficient of

$$a^k (\Sigma \alpha_1^{m_1} \alpha_2^{m_2} \dots \alpha_s^{m_s}).$$

We write this, usually, in the notation

$$a^k (m_1 m_2 \dots m_s).$$

The coefficient mentioned enumerates the partitions of the multipartite number

$$(m_1 m_2 \dots m_s),$$

into *k or fewer* parts. If the first fractional factor $1/1-a$ had been omitted the coefficient would have denoted the number of the partitions into *exactly* k parts. The inclusion of the factor $1/1-a$ is of great importance to the investigation and equally yields the enumerations into exactly k parts, because from the coefficients of $a^k (m_1 m_2 \dots m_s)$ we have merely to subtract the coefficients of $a^{k-1} (m_1 m_2 \dots m_s)$. The importance is due to the circumstance that the symmetric functions which present themselves in the expansion are in the best possible form for the performance of the Hammond operators. This is not the case when the factor $1/1-a$ is excluded, as then a transformation, the necessity for which is not at once clear, is needed to obtain the proper form.

I will remind the reader that, writing

$$(1-\alpha x)(1-\beta x)(1-\gamma x)\dots = 1 - a_1 x + a_2 x^2 - a_3 x^3 + \dots,$$

HAMMOND's differential operator of order m is

$$\text{D}_m = \frac{1}{m!}(\partial_{a_1} + a_1 \partial_{a_2} + a_2 \partial_{a_3} + \dots)^m ;$$

and its cardinal property is

$$D_{m_1}D_{m_2}\ldots D_{m_s}(m_1m_2\ldots m_s)=1\,;$$

and this operation does not result in unity when it is performed upon any other symmetric function.

In order to obtain the coefficient of

$$a^k(m_1m_2\ldots m_s),$$

in the expanded function, we first of all find the complete coefficient of a^k and then operate upon it with the Hammond combination of operators

$$D_{m_1}D_{m_2}\ldots D_{m_s}.$$

The result is an *integer* followed by the sum of an infinite series of symmetric functions. The integer mentioned is the number we seek.

Art. 5. We now expand the generating function. On well-known principles we can assert that the coefficient of a^k in the expansion is the *homogeneous product-sum of order k* of unity, and of the whole of the α, β, γ, ... products which occur in the denominator factors of the generating function. The *elements*, of which we must form homogeneous product sums are, in fact,

$$1$$

$$\alpha,\ \beta,\ \gamma,\ \ldots,$$

$$\alpha^2,\ \beta^2,\ \gamma^2,\ \ldots\ \alpha\beta,\ \alpha\gamma,\ \beta\gamma,\ \ldots,$$

$$\alpha^3,\ \beta^3,\ \gamma^3,\ \ldots\ \alpha^2\beta,\ \alpha^2\gamma,\ \ldots\alpha\beta\gamma,\ \alpha\beta\delta,\ \ldots.$$

. . . .

We can form these product-sums from the sums of the powers of these elements, because we have before us the well-known symmetric function formula

$$h_k=\Sigma_\sigma\frac{s_1^{k_1}s_2^{k_2}\ldots s_i^{k_i}}{1^{k_1}2^{k_2}\ldots i^{k_i}\,.\,k_1!\,k_2!\ldots k_i!}.$$

The sum of the powers are readily formed; for, calling them

$$\mathfrak{Q}_1,\ \mathfrak{Q}_2,\ \ldots\ \mathfrak{Q}_i,\ \ldots,$$

it is clear that $\mathfrak{Q}_i$ is the sum of unity and the whole of the monomial (that is to say merely involving in the partition notation a single partition), symmetric functions of weights one to infinity. Hence

$$\mathfrak{Q}_1=1+(1)+(2)+(1^2)+(3)+(21)+(1^3)+\ldots\text{ ad inf.}\,;$$

and, regarding Q_1 as $F(\alpha, \beta, \gamma, \ldots)$, it is obvious that

$$Q_i = F(\alpha^i, \beta^i, \gamma^i, \ldots);$$

showing us that

$$Q_2 = 1+(2)+(4)+(2^2)+(6)+(42)+(2^3)+\ldots,$$

$$Q_i = 1+(i)+(2i)+(i^2)+(3i)+(2i, i)+(i^3)+\ldots.$$

Thence the expansion

$$\begin{aligned}&1+\alpha Q_1\\ &+\frac{\alpha^2}{2\,!}(Q_1{}^2+Q_2)\\ &+\frac{\alpha^3}{3\,!}(Q_1{}^3+3Q_1Q_2+2Q_3)\\ &+\frac{\alpha^4}{4\,!}(Q_1{}^4+6Q_1{}^2Q_2+3Q_2{}^2+8Q_1Q_3+6Q_4)\\ &+\ldots\\ &+\alpha^k F_k(Q)\\ &+\ldots\end{aligned}$$

where

$$F_k(Q) = \Sigma \frac{Q_1{}^{k_1}Q_2{}^{k_2}\ldots Q_i{}^{k_i}}{1^{k_1}2^{k_2}\ldots i^{k_i}k_1\,!\,k_2\,!\ldots k_i\,!}.$$

Art. 6. The importance of this expansion lies in the fact that the infinite series of Hammond operators on the one hand and the infinite series of Q functions on the other hand have very remarkable properties in relation to one another. The first property we notice is that from the well-known law of operation,

$$D_m Q_1 = Q_1,$$

for all values of m. Also

$$D_m Q_2 = Q_2, \text{ or zero,}$$

ascending as m is, or is not, a multiple of *two*. And generally

$$D_m Q_i = Q_i, \text{ or zero,}$$

according as m is, or is not, a multiple of i.

When D_m is performed upon a product of k separate functions, it operates through the medium of a number of operators associated with the compositions of m into k parts, zero being regarded as a part and D_0 being regarded as a symbol for unity. Thus the compositions of the number 4 into three parts being 400, 040, 004; 310, 301, 130, 103, 031, 013; 220, 202, 022; 211, 121, 112, the law of operation is as follows:—

$$\begin{aligned} D_4\mathrm{Q}_a\mathrm{Q}_b\mathrm{Q}_c = &(D_4\mathrm{Q}_a)(D_0\mathrm{Q}_b)(D_0\mathrm{Q}_c)+(D_0\mathrm{Q}_a)(D_4\mathrm{Q}_b)(D_0\mathrm{Q}_c)+(D_0\mathrm{Q}_a)(D_0\mathrm{Q}_b)(D_4\mathrm{Q}_c)\\ &+(D_3\mathrm{Q}_a)(D_1\mathrm{Q}_b)(D_0\mathrm{Q}_c)+(D_3\mathrm{Q}_a)(D_0\mathrm{Q}_b)(D_1\mathrm{Q}_c)+(D_1\mathrm{Q}_a)(D_3\mathrm{Q}_b)(D_0\mathrm{Q}_c)\\ &+(D_1\mathrm{Q}_a)(D_0\mathrm{Q}_b)(D_3\mathrm{Q}_c)+(D_0\mathrm{Q}_a)(D_3\mathrm{Q}_b)(D_1\mathrm{Q}_c)+(D_0\mathrm{Q}_a)(D_1\mathrm{Q}_b)(D_3\mathrm{Q}_c)\\ &+(D_2\mathrm{Q}_a)(D_2\mathrm{Q}_b)(D_0\mathrm{Q}_c)+(D_2\mathrm{Q}_a)(D_0\mathrm{Q}_b)(D_2\mathrm{Q}_c)+(D_0\mathrm{Q}_a)(D_2\mathrm{Q}_b)(D_2\mathrm{Q}_c)\\ &+(D_2\mathrm{Q}_a)(D_1\mathrm{Q}_b)(D_1\mathrm{Q}_c)+(D_1\mathrm{Q}_a)(D_2\mathrm{Q}_b)(D_1\mathrm{Q}_c)+(D_1\mathrm{Q}_a)(D_1\mathrm{Q}_b)(D_2\mathrm{Q}_c) \end{aligned}$$

D_0, being unity, may be omitted but has been retained above to make the connexion with the compositions quite clear. This method of performing D_m upon a product I have explained and used in previous papers during the last five and twenty years. Upon this example some observations can be made. In the first place the operation breaks up into 15 portions because the number 4 has 15 three-part compositions. The result of each portion must be moreover either $\mathrm{Q}_a\mathrm{Q}_b\mathrm{Q}_c$ or zero, because $D_t\mathrm{Q}_s$ is either Q_s or zero. Hence the result of the whole operation must be merely to multiply $\mathrm{Q}_a\mathrm{Q}_b\mathrm{Q}_c$ by some integer $\leqq 15$. In general the result of the operation

$$D_m\mathrm{Q}_1^{k_1}\mathrm{Q}_2^{k_2}\ldots\mathrm{Q}_i^{k_i}$$

must be merely to multiply the product

$$\mathrm{Q}_1^{k_1}\mathrm{Q}_2^{k_2}\ldots\mathrm{Q}_i^{k_i}$$

by some integer equal to, or less, than the number of compositions of m into $k_1+k_2+\ldots+k_i$ parts, zero always counting as a part.

Hence also the result of the operation

$$D_{m_1}D_{m_2}\ldots D_{m_s}\mathrm{Q}_1^{k_1}\mathrm{Q}_2^{k_2}\ldots\mathrm{Q}_i^{k_i}$$

must be merely to multiply the product

$$\mathrm{Q}_1^{k_1}\mathrm{Q}_2^{k_2}\ldots\mathrm{Q}_i^{k_i}$$

by an integer.

This valuable result shows that the Hammond operators may be performed with facility upon the function

$$F_k(\mathrm{Q})$$

which is before us.

Art 7. The determination of the result of the operation

$$D_m\mathrm{Q}_1^{k_1}\mathrm{Q}_2^{k_2}\ldots\mathrm{Q}_i^{k_i}$$

is now entered upon. We have to find the value of the multiplying integer

The value is clearly equal to the number of compositions of m which do not have the effect of multiplication by zero. Suppose that we write out the operand at length

$$\mathbf{Q}_1\mathbf{Q}_1\mathbf{Q}_1 \ldots \mathbf{Q}_2\mathbf{Q}_2\mathbf{Q}_2 \ldots\ldots \mathbf{Q}_i\mathbf{Q}_i\mathbf{Q}_i \ldots$$

in i blocks, containing $k_1, k_2, \ldots k_i$ factors respectively.

Underneath the k factors we will suppose written any composition of m into k parts, zero being included as a part.

In order that the corresponding operation may result in unity and not in zero we have the conditions :—

(i.) Any number, including zero, may occur underneath any of the k_1 factors $\mathbf{Q}_1$;

(ii.) Zero, or any multiple of 2, may occur underneath any of the k_2 factors $\mathbf{Q}_2$;

(iii.) Zero, or any multiple of 3, may occur underneath any of the k_3 factors $\mathbf{Q}_3$;

(iii. ...) And lastly, zero, or any multiple of i, may occur underneath any of the k_i factors $\mathbf{Q}_i$.

How many such compositions exist ?

We have merely to find the coefficient of x^m in the expansion of

$$(1+x+x^2+\ldots)^{k_1}(1+x^2+x^4+\ldots)^{k_2}(1+x^3+x^6+\ldots)^{k_3}\ldots\ldots(1+x^i+x^{2i}+\ldots)^{k_i};$$

for this is the function which enumerates the compositions which possess these properties. In fact to form the composition we take a power of x from each of the first k_1 factors ; then a power of x from each of the next k_2 factors, observing that the exponents of x are all zero or multiples of two ; then a power of x from the next k_3 factors, observing that the exponents are all zero or multiples of three ; and so on, until finally in the k_i factors we find that the exponents are all zero or multiples of i.

Hence it follows that the operation

$$\mathrm{D}_m\mathbf{Q}_1^{k_1}\mathbf{Q}_2^{k_2}\ldots\mathbf{Q}_i^{k_i}$$

has the effect of multiplying $\mathbf{Q}_1^{k_1}\mathbf{Q}_2^{k_2}\ldots\mathbf{Q}_i^{k_i}$ by a number which is given by the coefficient of x^m in the expansion of

$$(1-x)^{-k_1}(1-x^2)^{-k_2}(1-x^3)^{-k_3}\ldots(1-x^i)^{-k_i};$$

an elegant theorem.

Let this coefficient be denoted by

$$\mathrm{F}_q(m;\ 1^{k_1}2^{k_2}\ldots i^{k_i});$$

so that

$$\mathrm{D}_m\mathbf{Q}_1^{k_1}\mathbf{Q}_2^{k_2}\ldots\mathbf{Q}_i^{k_i} = \mathrm{F}_q(m;\ 1^{k_1}2^{k_2}\ldots i^{k_i})\ .\ \mathbf{Q}_1^{k_1}\mathbf{Q}_2^{k_2}\ldots\mathbf{Q}_i^{k_i}$$

Art. 8. Looking to the symmetric function expressions of $\mathbf{Q}_1, \mathbf{Q}_2, \ldots \mathbf{Q}_i$ it will be noted that the only portion of the product

$$\mathbf{Q}_1^{k_1}\mathbf{Q}_2^{k_2}\ldots\mathbf{Q}_i^{k_i}$$

which does not involve the elements $\alpha, \beta, \gamma, \ldots$ is unity.

Hence the portion of

$$D_m \mathbf{Q}_1^{k_1} \mathbf{Q}_2^{k_2} \ldots \mathbf{Q}_i^{k_i}$$

that is free from the elements is

$$F_q(m;\ 1^{k_1} 2^{k_2} \ldots i^{k_i}),$$

which is obtained directly from the result of the operation by putting

$$\mathbf{Q}_1 = \mathbf{Q}_2 = \ldots = \mathbf{Q}_i = 1.$$

We may represent this circumstance by the convenient notation

$$(D_m \mathbf{Q}_1^{k_1} \mathbf{Q}_2^{k_2} \ldots \mathbf{Q}_i^{k_i})_{Q=1} = F_q(m;\ 1^{k_1} 2^{k_2} \ldots i^{k_i}).$$

The number of partitions, of the unipartite number m, into k or fewer parts, is by the present investigation

$$D_m F_k(\mathbf{Q})_{Q=1}$$

$$= \Sigma \frac{(D_m \mathbf{Q}_1^{k_1} \mathbf{Q}_2^{k_2} \ldots \mathbf{Q}_i^{k_i})_{Q=1}}{1^{k_1} 2^{k_2} \ldots i^{k_i} \,.\, k_1!\, k_2! \ldots k_i!}$$

$$= \Sigma \frac{F_q(m;\ 1^{k_1} 2^{k_2} \ldots i^{k_i})}{1^{k_1} 2^{k_2} \ldots i^{k_i} \,.\, k_1!\, k_2! \ldots k_i!}$$

$$= \text{coefficient of } x^m \text{ in } \Sigma \frac{(1-x)^{-k_1} (1-x^2)^{-k_2} \ldots (1-x^i)^{-k_i}}{1^{k_1} 2^{k_2} \ldots i^{k_i} \,.\, k_1!\, k_2! \ldots k_i!};$$

the summation being for every partition

$$k_1 + 2k_2 + \ldots + ik_i$$

of the number k.

But we know, otherwise, that the number of partitions of m, into k or fewer parts, is given by the coefficients of x^m in

$$\frac{1}{(1-x)(1-x^2) \ldots (1-x^k)}.$$

Hence the identity

$$\Sigma \frac{(1-x)^{-k_1} (1-x^2)^{-k_2} \ldots (1-x^i)^{-k_i}}{1^{k_1} 2^{k_2} \ldots i^{k_i} k_1!\, k_2! \ldots k_i!} = \frac{1}{(1-x)(1-x^2) \ldots (1-x^k)},$$

which being a known result supplies an interesting verification of our work. The present investigation in any case supplies one proof of it.

Art. 9. There is now no difficulty in proceeding to the result

$$D_{m_1} D_{m_2} \ldots D_{m_s} \,.\, \mathbf{Q}_1^{k_1} \mathbf{Q}_2^{k_2} \ldots \mathbf{Q}_i^{k_i}$$

$$= F_q(m_1;\ 1^{k_1} 2^{k_2} \ldots i^{k_i}) \,.\, F_q(m_2;\ 1^{k_1} 2^{k_2} \ldots i^{k_i}) \ldots\ldots F_q(m_s;\ 1^{k_1} 2^{k_2} \ldots i^{k_i}) \,.\, \mathbf{Q}_1^{k_1} \mathbf{Q}_2^{k_2} \ldots \mathbf{Q}_i^{k_i};$$

and the number of partitions of the multipartite number

$$m_1 m_2 \dots m_s,$$

into k or fewer parts, is

$$\mathrm{D}_{m_1}\mathrm{D}_{m_2}\dots\mathrm{D}_{m_s}\mathrm{F}_k(\mathbf{Q})_{\mathrm{Q}=1}$$

$$= \Sigma \frac{(\mathrm{D}_{m_1}\mathrm{D}_{m_2}\dots\mathrm{D}_{m_s}\mathbf{Q}_1^{k_1}\mathbf{Q}_2^{k_2}\dots\mathbf{Q}_i^{k_i})_{\mathrm{Q}=1}}{1^{k_1}2^{k_2}\dots i^{k_i}.\,k_1\,!\,k_2\,!\dots k_i\,!}$$

$$= \Sigma \frac{\mathrm{F}_q(m_1;\ 1^{k_1}2^{k_2}\dots i^{k_i}).\,\mathrm{F}_q(m_2;\ 1^{k_1}2^{k_2}\dots i^{k_i})\dots\dots\mathrm{F}_q(m_s;\ 1^{k_1}2^{k_2}\dots i^{k_i})}{1^{k_1}2^{k_2}\dots i^{k_i}.\,k_1\,!\,k_2\,!\dots k_i\,!}$$

This is the general solution of the problem of enumeration in the absence of any restriction upon the magnitudes of the constituents of the multipartite parts.*

Art. 10. It will be convenient at this point to give a few results derived from the function

$$(1-x)^{-k_1}(1-x^2)^{-k_2}\dots(1-x^i)^{-k_i}$$

which will be useful in the sequel.

$$\mathrm{F}_q(m;\ 1^k) = \binom{m+k-1}{k-1},$$

$$\mathrm{F}_q(2m;\ 2^k) = \binom{m+k-1}{k-1},$$

$$\mathrm{F}_q(im;\ i^k) = \binom{m+k-1}{k-1},$$

$$\mathrm{F}_q(2m;\ 12) = m+1, \qquad \mathrm{F}_q(2m+1;\ 12) = m+1,$$

$$\mathrm{F}_q(2m;\ 1^2 2) = (m+1)^2, \qquad \mathrm{F}_q(2m+1;\ 1^2 2) = (m+1)(m+2),$$

$$\mathrm{F}_q(2m;\ 12^2) = \binom{m+2}{2}, \qquad \mathrm{F}_q(2m+1;\ 12^2) = \binom{m+2}{2},$$

* With regard to the algebraical identity met with above, the reader may compare 'SYLVESTER'S Mathematical Papers,' vol. III., p. 598, where it is shown that for the roots of the equation

$$z^q - \frac{1}{1-c}z^{q-1} + \frac{c}{(1-c)(1-c^2)}z^{q-2} - \frac{c^3}{(1-c)(1-c^2)(1-c^3)}z^{q-3} + \dots = 0,$$

the general term being

$$(-)^n \frac{c^{\binom{n}{2}}}{(1-c)(1-c^2)\dots(1-c^n)} z^{q-n},$$

the homogeneous product-sum of order n is

$$\frac{1}{(1-c)(1-c^2)\dots(1-c^n)};$$

and the sum of the n^{th} powers of the roots is

$$\frac{1}{1-c^n}.$$

The expression of the homogeneous product-sum of order k, in terms of the sums of the powers, by the formula quoted early in this paper, gives the identity in question.

$$F_q(3m;\ 13) = F_q(3m+1;\ 13) = F_q(3m+2;\ 13) = m+1,$$

$$F_q(3m;\ 1^2 3) = \tfrac{1}{2}(m+1)(3m+2), \qquad F_q(3m+1;\ 1^2 3) = \tfrac{1}{2}(m+1)(3m+4),$$

$$F_q(3m+2;\ 1^2 3) = \tfrac{1}{2}(m+1)(3m+6),$$

$$F_q(2m;\ 1^{k_1}2^{k_2})$$
$$= \binom{m+k_2-1}{k_2-1} + \binom{k_1+1}{k_1-1}\binom{m+k_2-2}{k_2-1} + \binom{k_1+3}{k_1-1}\binom{m+k_2-3}{k_2-1} + \ldots + \binom{2m+k_1-1}{k_1-1},$$

$$F_q(2m+1;\ 1^{k_1}2^{k_2})$$
$$= \binom{k_1}{k_1-1}\binom{m+k_2-1}{k_2-1} + \binom{k_1+2}{k_1-1}\binom{m+k_2-2}{k_2-1} + \ldots + \binom{2m+k_1}{k_1-1},$$

$$F_q(3m;\ 1^{k_1}3^{k_3})$$
$$= \binom{m+k_3-1}{k_3-1} + \binom{k_1+2}{k_1-1}\binom{m+k_3-2}{k_3-1} + \binom{k_1+5}{k_1-1}\binom{m+k_3-3}{k_3-1} + \ldots + \binom{3m+k_1-1}{k_1-1},$$

$$F_q(3m+1;\ 1^{k_1}3^{k_3})$$
$$= \binom{k_1}{k_1-1}\binom{m+k_3-1}{k_3-1} + \binom{k_1+3}{k_1-1}\binom{m+k_3-2}{k_3-1} + \ldots + \binom{3m+k_1}{k_1-1},$$

$$F_q(3m+2;\ 1^{k_1}3^{k_3})$$
$$= \binom{k_1+1}{k_1-1}\binom{m+k_3-1}{k_3-1} + \binom{k_1+4}{k_1-1}\binom{m+k_3-2}{k_3-1} + \ldots + \binom{3m+k_1+1}{k_1-1}.$$

The use of the Hammond operator D_m is convenient but not essential to this investigation. It is convenient from the algebraic point of view, and also because it brings into prominence properties of the operator which are in themselves important. The coefficient of $\alpha_1^{m_1}\alpha_2^{m_2}\ldots\alpha_s^{m_s}$ in the product $\mathfrak{Q}_1^{k_1}\mathfrak{Q}_2^{k_2}\ldots\mathfrak{Q}_i^{k_i}$ is readily obtained when we remember that

$$\mathfrak{Q}_i = \frac{1}{(1-\alpha^i)(1-\beta^i)(1-\gamma^i)\ldots\ldots}$$

and the various modifications are readily made for the allied functions A_i, B_i, ... U_i.

The Partitions of Multipartite Numbers into Two Parts.

Art. 11. The generating function which enumerates the partitions into two or fewer parts is

$$\tfrac{1}{2}(\mathfrak{Q}_1^{\,2}+\mathfrak{Q}_2);$$

and since, from the principles just stated,

$$D_{2m}\mathfrak{Q}_1^{\,2} = (2m+1)\,\mathfrak{Q}_1^{\,2}, \qquad D_{2m+1}\mathfrak{Q}_1^{\,2} = (2m+2)\,\mathfrak{Q}_1^{\,2},$$

$$D_{2m}\mathfrak{Q}_2 = \mathfrak{Q}_2, \qquad D_{2m+1}\mathfrak{Q}_2 = 0;$$

we find

$$D_{2m}\tfrac{1}{2}(\mathbf{Q}_1{}^2+\mathbf{Q}_2) = \tfrac{1}{2}(2m+1)\,\mathbf{Q}_1{}^2+\tfrac{1}{2}\mathbf{Q}_2,$$

$$D_{2m+1}\tfrac{1}{2}(\mathbf{Q}_1{}^2+\mathbf{Q}_2) = \tfrac{1}{2}(2m+2)\,\mathbf{Q}_1{}^2\,;$$

and, by reason of the important properties possessed by the $\mathbf{Q}$ products in their relations with the Hammond operators, we can at once proceed to the results

$$D^s_{2m} = \tfrac{1}{2}(2m+1)^s\mathbf{Q}_1{}^2+\tfrac{1}{2}\mathbf{Q}_2,$$

$$D^s_{2m+1} = \tfrac{1}{2}(2m+2)^s\mathbf{Q}_1{}^2.$$

Thence we derive, by putting $\mathbf{Q}_1 = \mathbf{Q}_2 = 1$, the coefficients of the symmetric functions

$$(2m^s), \qquad (2m+1^s)$$

(the exponent s meaning the numbers $2m$, $2m+1$ respectively s times repeated) in the development of the function

$$\tfrac{1}{2}(\mathbf{Q}_1{}^2+\mathbf{Q}_2).$$

Thence we obtain the numbers

$$\tfrac{1}{2}(2m+1)^s+\tfrac{1}{2}, \qquad \tfrac{1}{2}(2m+2)^s$$

which, respectively, enumerate the ways of partitioning the *multipartite numbers*

$$(2m,\ 2m \ldots \text{repeated } s \text{ times}), \qquad (2m+1,\ 2m+1 \ldots \text{repeated } s \text{ times})$$

into two or fewer parts.

When the enumeration is concerned with exactly two parts we have clearly to subtract unity in each case. In fact the generating function is

$$\tfrac{1}{2}(\mathbf{Q}_1{}^2+\mathbf{Q}_2)-\mathbf{Q}_1\,;$$

and

$$D^s_{2m}\mathbf{Q}_1 = D^s_{2m+1}\mathbf{Q}_1 = \mathbf{Q}_1,$$

showing that unity must be subtracted.

The numbers then become

$$\tfrac{1}{2}(2m+1)^s-\tfrac{1}{2}, \qquad \tfrac{1}{2}(2m+2)^s-1.$$

These numbers also enumerate the ways of exhibiting the composite integers

$$(p_1p_2\ldots p_s)^{2m}, \qquad (p_1p_2\ldots p_s)^{2m+1},$$

as the product of two factors.

To obtain a general formula for the multipartite number

$$(m_1m_2\ldots m_s)$$

we write

$$D_m \mathbf{Q}_1^2 = F_q(m;\ 1^2) . \mathbf{Q}_1^2,$$

$$D_m \mathbf{Q}_2 = F_q(m;\ 2) . \mathbf{Q}_2,$$

then

$$D_{m_1} \tfrac{1}{2} (\mathbf{Q}_1^2 + \mathbf{Q}_2) = \tfrac{1}{2} \{F_q(m_1;\ 1^2) . \mathbf{Q}_1^2 + F_q(m_1;\ 2) . \mathbf{Q}_2\},$$

$$D_{m_1} D_{m_2} \tfrac{1}{2} (\mathbf{Q}_1^2 + \mathbf{Q}_2) = \tfrac{1}{2} \{F_q(m_1;\ 1^2)\, F_q(m_2;\ 1^2) . \mathbf{Q}_1^2 + F_q(m_1;\ 2)\, F_q(m_2;\ 2) . \mathbf{Q}_2\},$$

$$D_{m_1} D_{m_2} \ldots D_{m_s} \tfrac{1}{2} (\mathbf{Q}_1^2 + \mathbf{Q}_2) = \tfrac{1}{2} \prod_1^s F_q(m_i;\ 1^2) . \mathbf{Q}_1^2 + \tfrac{1}{2} \prod_1^s F_q(m_i;\ 2) . \mathbf{Q}_2,$$

leading us to the number

$$\tfrac{1}{2} \prod_1^s F_q(m_i;\ 1^2) + \tfrac{1}{2} \prod_1^s F_q(m_i;\ 2),$$

as the enumerator of the partitions, into two or fewer parts, of the multipartite number

$$(m_1 m_2 \ldots m_s).$$

The reader will observe that the algebraic expressions of

$$F_q(m_i;\ 1^2) \text{ and } F_q(m_i;\ 2)$$

will depend upon the parity of m_i.

The notation has been adopted so as to save a multiplicity of formulæ in certain cases.

This of course solves the question of the factorization into two or fewer factors of the composite integer

$$p_1^{m_1} p_2^{m_2} \ldots p_s^{m_s}.$$

We have, therefore, solved completely the question of enumerating the bipartitions of multipartite numbers.

What has been done as a question in the theory of distribution may be stated as follows. We are given an assemblage of any numerical specification and two boxes which cannot be distinguished from one another. We have found the number of ways of distributing the objects between the boxes. The similar question when the boxes are distinguished from one another is simpler and connected with the compositions of multipartite numbers.

Art. 12. At this point it may be appropriate to give a statement in regard to the nature of the solution given in this investigation.

The enumeration of the partitions of a unipartite number m_1, into k or fewer parts, is formed as a linear function of certain numbers $a_1, b_1, c_1 \ldots$; the linear function being

$$\lambda a_1 + \mu b_1 + \nu c_1 + \ldots$$

where the numbers λ, μ, ν, depend only upon k.

Associated with another unipartite number m_2 we have the linear function

$$\lambda a_2 + \mu b_2 + \nu c_2 + \ldots .$$

It has then been shown that the number of partitions, into k or fewer parts, of the multipartite number $m_1 m_2$ is

$$\lambda a_1 a_2 + \mu b_1 b_2 + \nu c_1 c_2 + \ldots ,$$

and in general the number of partitions, into k or fewer parts, of the multipartite number

$$m_1 m_2 \ldots m_s$$

is

$$\lambda a_1 a_2 \ldots a_s + \mu b_1 b_2 \ldots b_s + \nu c_1 c_2 \ldots c_s + \ldots .$$

The multipartite solution is thus essentially derived from the solutions which appertain to the separate unipartite numbers whose conjunction defines the multipartite number. The numbers $\lambda, \mu, \nu, \ldots$ are those well known in connexion with the expression of the homogeneous product sum h_k in terms of the sums of the powers $s_1, s_2, s_3, \ldots s_k$, the whole question is therefore reduced to finding the numbers

$$a, b, c, \ldots$$

appertaining to the unipartite number m.

This, as has been shown, depends upon finding the coefficient of x^m in a function

$$(1-x)^{-k_1}(1-x^2)^{-k_2} \ldots (1-x^i)^{-k_i}$$

where

$$k_1 + k_2 + \ldots + k_i = k.$$

The possibility of the solution rests upon the remarkable circumstances that when the operator D_m is performed upon the operand

$$Q_1^{k_1} Q_2^{k_2} \ldots Q_i^{k_i}$$

its effect is to merely multiply it by an integer.

The Partitions of Multipartite Numbers into Three Parts.

Art. 13. I will, in future, merely deal with the partitions into k or fewer parts, since the result for exactly k parts is at once derived by subtracting the result for $k-1$ or fewer parts.

The operand is

$$\tfrac{1}{6}(Q_1^3 + 3Q_1Q_2 + 2Q_3),$$

and since the result depends upon the divisibility of m by both 2 and 3 it will be necessary to consider the operations of

$$D_{6m}, D_{6m+1}, D_{6m+2}, D_{6m+3}, D_{6m+4}, D_{6m+5}.$$

The investigation is therefore in six parts.

(i.) Since

$$D_{6m}\mathbf{Q}_1^3 = \binom{6m+2}{2}\mathbf{Q}_1^3,$$

$$D_{6m}\mathbf{Q}_1\mathbf{Q}_2 = (3m+1)\mathbf{Q}_1\mathbf{Q}_2,$$

$$D_{6m}\mathbf{Q}_3 = \mathbf{Q}_3,$$

we find

$$D_{6m}\tfrac{1}{6}(\mathbf{Q}_1^3+3\mathbf{Q}_1\mathbf{Q}_2+2\mathbf{Q}_3) = \tfrac{1}{6}\left\{\binom{6m+2}{2}\mathbf{Q}_1^3+3(3m+1)\mathbf{Q}_1\mathbf{Q}_2+2\mathbf{Q}_3\right\};$$

and immediately

$$D_{6m_1}D_{6m_2}\ldots D_{6m_s}\tfrac{1}{6}(\mathbf{Q}_1^3+3\mathbf{Q}_1\mathbf{Q}_2+2\mathbf{Q}_3) = \tfrac{1}{6}\left\{\binom{6m_1+2}{2}\binom{6m_2+2}{2}\ldots\binom{6m_s+2}{2}\mathbf{Q}_1^3\right.$$
$$+3(3m_1+1)(3m_2+1)\ldots(3m_s+1)\mathbf{Q}_1\mathbf{Q}_2$$
$$\left.+2\mathbf{Q}_3\right\};$$

and in particular

$$D_{6m}^s\tfrac{1}{6}(\mathbf{Q}_1^3+3\mathbf{Q}_1\mathbf{Q}_2+2\mathbf{Q}_3) = \tfrac{1}{6}\left\{\binom{6m+2}{2}^s\mathbf{Q}_1^3+3(3m+1)^s\mathbf{Q}_1\mathbf{Q}_2+2\mathbf{Q}_3\right\};$$

results which establish that the partitions of the multipartite number

$$6m_1 6m_2 \ldots 6m_s,$$

into three or fewer parts, are enumerated by

$$\tfrac{1}{6}\left\{\binom{6m_1+2}{2}\binom{6m_2+2}{2}\ldots\binom{6m_s+2}{2}+3(3m_1+1)(3m_2+1)\ldots(3m_s+1)+2\right\}.$$

(ii.) Since

$$D_{6m+1}\mathbf{Q}_1^3 = \binom{6m+3}{2}\mathbf{Q}_1^3,$$

$$D_{6m+1}\mathbf{Q}_1\mathbf{Q}_2 = (3m+1)\mathbf{Q}_1\mathbf{Q}_2,$$

$$D_{6m+1}\mathbf{Q}_3 = 0,$$

we find

$$D_{6m+1}\tfrac{1}{6}(\mathbf{Q}_1^3+3\mathbf{Q}_1\mathbf{Q}_2+2\mathbf{Q}_3) = \tfrac{1}{6}\left\{\binom{6m+3}{2}\mathbf{Q}_1^3+3(3m+1)\mathbf{Q}_1\mathbf{Q}_2\right\};$$

and immediately

$$D_{6m_1+1}D_{6m_2+1}\ldots D_{6m_s+1}\tfrac{1}{6}(\mathbf{Q}_1^3+3\mathbf{Q}_1\mathbf{Q}_2+2\mathbf{Q}_3) = \tfrac{1}{6}\left\{\binom{6m_1+3}{2}\binom{6m_2+3}{2}\ldots\binom{6m_s+3}{2}\mathbf{Q}_1^3\right.$$
$$\left.+3(3m_1+1)(3m_2+1)\ldots(3m_s+1)\mathbf{Q}_1\mathbf{Q}_2\right\};$$

establishing that the partitions of the multipartite number

$$6m_1+1\ \ 6m_2+1\ldots 6m_s+1,$$

into three or fewer parts are enumerated by

$$\tfrac{1}{6}\left\{\binom{6m_1+3}{2}\binom{6m_2+3}{2}\ldots\binom{6m_s+3}{2}+3\,(3m_1+1)\,(3m_2+1)\ldots(3m_s+1)\right\}.$$

(iii.) Since

$$D_{6m+2}\mathbf{Q}_1^{\,3} = \binom{6m+4}{2}\mathbf{Q}_1^{\,3},$$

$$D_{6m+2}\mathbf{Q}_1\mathbf{Q}_2 = (3m+2)\,\mathbf{Q}_1\mathbf{Q}_2,$$

$$D_{6m+2}\mathbf{Q}_3 = 0,$$

we, as above, derive, for the partitions of the multipartite number

$$6m_1+2\ \ 6m_2+2\ \ldots\ 6m_s+2,$$

the enumerating number

$$\tfrac{1}{6}\left\{\binom{6m_1+4}{2}\binom{6m_2+4}{2}\ldots\binom{6m_s+4}{2}+3\,(3m_1+2)\,(3m_2+2)\ldots(3m_s+2)\right\}.$$

(iv.) Since

$$D_{6m+3}\mathbf{Q}_1^{\,3} = \binom{6m+5}{2}\mathbf{Q}_1^{\,3},$$

$$D_{6m+3}\mathbf{Q}_1\mathbf{Q}_2 = (3m+2)\,\mathbf{Q}_1\mathbf{Q}_2,$$

$$D_{6m+3}\mathbf{Q}_3 = \mathbf{Q}_3,$$

we obtain, for the multipartite number

$$6m_1+3\ \ 6m_2+3\ \ldots\ 6m_s+3,$$

the enumerating number

$$\tfrac{1}{6}\left\{\binom{6m_1+5}{2}\binom{6m_2+5}{2}\ldots\binom{6m_s+5}{2}+3\,(3m_1+2)\,(3m_2+2)\ldots(3m_s+2)+2\right\}.$$

(v.) Also

$$D_{6m+4}\mathbf{Q}_1^{\,3} = \binom{6m+6}{2}\mathbf{Q}_1^{\,3},$$

$$D_{6m+4}\mathbf{Q}_1\mathbf{Q}_2 = (3m+3)\,\mathbf{Q}_1\mathbf{Q}_2,$$

$$D_{6m+4}\mathbf{Q}_3 = 0,$$

and we obtain, for the multipartite number

$$6m_1+4\ \ 6m_2+4\ \ldots\ 6m_s+4,$$

the enumerating number

$$\tfrac{1}{6}\left\{\binom{6m_1+6}{2}\binom{6m_2+6}{2}\ldots\binom{6m_s+6}{2}+3\,(3m_1+3)\,(3m_2+3)\ldots(3m_s+3)\right\}.$$

(vi.) Lastly, since

$$D_{6m+5}Q_1^{\,3}=\binom{6m+7}{2}Q_1^{\,3},$$

$$D_{6m+5}Q_1Q_2=(3m+3)\,Q_1Q_2,$$

$$D_{6m+5}Q_3=0,$$

we obtain, for the multipartite number

$$6m_1+5\;\;6m_2+5\;\ldots\;6m_s+5,$$

the enumerating number

$$\tfrac{1}{6}\left\{\binom{6m_1+7}{2}\binom{6m_2+7}{2}\ldots\binom{6m_s+7}{2}+3\,(3m_1+3)\,(3m_2+3)\ldots(3m_s+3)\right\}.$$

Finally, in the notation employed for the bipartite case, for the multipartite number

$$m_1m_2\ldots m_s$$

we have the enumerating number

$$\tfrac{1}{6}\left\{\prod_1^s F_q(m_i;\;1^3)+3\prod_1^s F_q(m_i;\;1,2)+2\prod_1^s F_q(m_i;\;3)\right\}.$$

Art. 14. I collect these results:—

Multipartite Numbers.	Number of Partitions into three or fewer parts.
$6m\;6m\;6m$ repeated s times	$\frac{1}{3!}\left\{\binom{6m+2}{2}^s+3\,(3m+1)^s+2\right\}$
$6m+1\;6m+1\;6m+1$ repeated s times	$\frac{1}{3!}\left\{\binom{6m+3}{2}^s+3\,(3m+1)^s\right\}$
$6m+2\;6m+2\;6m+2$,, ,,	$\frac{1}{3!}\left\{\binom{6m+4}{2}^s+3\,(3m+2)^s\right\}$
$6m+3\;6m+3\;6m+3$,, ,,	$\frac{1}{3!}\left\{\binom{6m+5}{2}^s+3\,(3m+2)^s+2\right\}$
$6m+4\;6m+4\;6m+4$,, ,,	$\frac{1}{3!}\left\{\binom{6m+6}{2}^s+3\,(3m+3)^s\right\}$
$6m+5\;6m+5\;6m+5$,, ,,	$\frac{1}{3!}\left\{\binom{6m+7}{2}^s+3\,(3m+3)^s\right\}.$

Multipartite Numbers.	Number of Partitions into exactly three parts.
$6m\ 6m\ 6m$ repeated s times	$\frac{1}{3!}\left\{\binom{6m+2}{2}^s+3(3m+1)^s-3(6m+1)^s-1\right\}$
$6m+1\ 6m+1\ 6m+1$ repeated s times	$\frac{1}{3!}\left\{\binom{6m+3}{2}^s+3(3m+1)^s-3(6m+2)^s\right\}$
$6m+2\ 6m+2\ 6m+2$,, ,,	$\frac{1}{3!}\left\{\binom{6m+4}{2}^s+3(3m+2)^s-3(6m+3)^s-3\right\}$
$6m+3\ 6m+3\ 6m+3$,, ,,	$\frac{1}{3!}\left\{\binom{6m+5}{2}^s+3(3m+2)^s-3(6m+4)^s+2\right\}$
$6m+4\ 6m+4\ 6m+4$,, ,,	$\frac{1}{3!}\left\{\binom{6m+6}{2}^s+3(3m+3)^s-3(6m+5)^s-3\right\}$
$6m+5\ 6m+5\ 6m+5$,, ,,	$\frac{1}{3!}\left\{\binom{6m+7}{2}^s+3(3m+3)^s-3(6m+6)^s\right\}$.

As a verification, connected with unipartite partitions, we put $s = 1$ in these last six formulæ, and reach the six numbers

$$3m^2,\quad 3m^2+m,\quad 3m^2+2m,\quad 3m^2+3m+1,\quad 3m^2+4m+1,\quad 3m^2+5m+2,$$

and since these may be exhibited in the forms

$$\frac{(6m)^2}{12},\quad \frac{(6m+1)^2}{12}-\tfrac{1}{12},\quad \frac{(6m+2)^2}{12}-\tfrac{1}{3},\quad \frac{(6m+3)^2}{12.}+\tfrac{1}{4},\quad \frac{(6m+4)^2}{12}-\tfrac{1}{3},\quad \frac{(6m+5)^2}{12}-\tfrac{1}{12},$$

we verify the well-known theorem which states that the number of tripartitions of n is the nearest integer to $\frac{n^2}{12}$.

The Partitions of Multipartite Numbers into Four Parts.

Art. 15. The operand is

$$\tfrac{1}{24}\left(\mathbf{Q}_1{}^4+6\mathbf{Q}_1{}^2\mathbf{Q}_2+3\mathbf{Q}_2{}^2+8\mathbf{Q}_1\mathbf{Q}_3+6\mathbf{Q}_4\right)$$

since the result depends upon the divisibility of m by the numbers 2, 3 and 4, and 12 is the least common multiple of those numbers, it will be necessary to take the operator suffix to the modulus 12, and the investigation is, therefore, in twelve parts.

We have

$$
\begin{aligned}
D_m \mathbf{Q}_1^4 &= \binom{m+3}{3} \mathbf{Q}_1^4,\\
D_{2m} \mathbf{Q}_1^2 \mathbf{Q}_2 &= (m+1)^2 \mathbf{Q}_1^2 \mathbf{Q}_2,\\
D_{2m+1} \mathbf{Q}_1^2 \mathbf{Q}_2 &= (m+1)(m+2)\, \mathbf{Q}_1^2 \mathbf{Q}_2,\\
D_{2m} \mathbf{Q}_2^2 &= (m+1)\, \mathbf{Q}_2^2,\\
D_{2m+1} \mathbf{Q}_2^2 &= 0,\\
D_{3m} \mathbf{Q}_1 \mathbf{Q}_3 &= (m+1) \mathbf{Q}_1 \mathbf{Q}_3,\\
D_{3m+1} \mathbf{Q}_1 \mathbf{Q}_3 &= (m+1)\, \mathbf{Q}_1 \mathbf{Q}_3,\\
D_{3m+2} \mathbf{Q}_1 \mathbf{Q}_3 &= (m+1)\, \mathbf{Q}_1 \mathbf{Q}_3,\\
D_{4m} \mathbf{Q}_4 &= \mathbf{Q}_4,\\
D_{4m+1} \mathbf{Q}_4 &= D_{4m+2} \mathbf{Q}_4 = D_{4m+3} \mathbf{Q}_4 = 0.
\end{aligned}
$$

Utilising these results and taking as operators D^s_{12m}, D^s_{12m+1}, … D^s_{12m+11} in succession we find for partitions into four or fewer parts :—

Multipartite Numbers.	Number of Partitions into four or fewer parts.
$12m$ $12m$ $12m$ repeated s times	$\frac{1}{4!}\left\{\binom{12m+3}{3}^s + 6(6m+1)^{2s} + 3(6m+1)^s + 8(4m+1)^s + 6\right\}$
$12m+1$ $12m+1$ $12m+1$ repeated s times	$\frac{1}{4!}\left\{\binom{12m+4}{3}^s + 12\binom{6m+2}{2}^s + 8(4m+1)^s\right\}$
$12m+2$ $12m+2$ $12m+2$,, ,,	$\frac{1}{4!}\left\{\binom{12m+5}{3}^s + 6(6m+2)^{2s} + 3(6m+2)^s + 8(4m+1)^s\right\}$
$12m+3$ $12m+3$ $12m+3$,, ,,	$\frac{1}{4!}\left\{\binom{12m+6}{3}^s + 12\binom{6m+3}{2}^s + 8(4m+2)^s\right\}$
$12m+4$ $12m+4$ $12m+4$,, ,,	$\frac{1}{4!}\left\{\binom{12m+7}{3}^s + 6(6m+3)^{2s} + 3(6m+3)^s + 8(4m+2)^s + 6\right\}$
$12m+5$ $12m+5$ $12m+5$,, ,,	$\frac{1}{4!}\left\{\binom{12m+8}{3}^s + 12\binom{6m+4}{2}^s + 8(4m+2)^s\right\}$
$12m+6$ $12m+6$ $12m+6$,, ,,	$\frac{1}{4!}\left\{\binom{12m+9}{3}^s + 6(6m+4)^{2s} + 3(6m+4)^s + 8(4m+3)^s\right\}$
$12m+7$ $12m+7$ $12m+7$,, ,,	$\frac{1}{4!}\left\{\binom{12m+10}{3}^s + 12\binom{6m+5}{2}^s + 8(4m+3)^s\right\}$
$12m+8$ $12m+8$ $12m+8$,, ,,	$\frac{1}{4!}\left\{\binom{12m+11}{3}^s + 6(6m+5)^{2s} + 3(6m+5)^s + 8(4m+3)^s + 6\right\}$
$12m+9$ $12m+9$ $12m+9$,, ,,	$\frac{1}{4!}\left\{\binom{12m+12}{3}^s + 12\binom{6m+6}{2}^s + 8(4m+4)^s\right\}$
$12m+10$ $12m+10$ $12m+10$,, ,,	$\frac{1}{4!}\left\{\binom{12m+13}{3}^s + 6(6m+6)^{2s} + 3(6m+6)^s + 8(4m+4)^s\right\}$
$12m+11$ $12m+11$ $12m+11$,, ,,	$\frac{1}{4!}\left\{\binom{12m+14}{3}^s + 12\binom{6m+7}{2}^s + 8(4m+4)^s\right\}$

For the unipartite case we put $s = 1$ and find, reading by rows, the numbers

m^3.	m^2,	m.	1.
12	15	6	1
12	18	8	1
12	21	12	2
12	24	15	3
12	27	20	5
12	30	24	6
12	33	30	9
12	36	35	11
12	39	42	15
12	42	48	18
12	45	56	23
12	48	63	27

which admit of easy verification.

In the notation of this paper, for the multipartite number

$$m_1 m_2 \dots m_s,$$

we have the enumerating number

$$\tfrac{1}{24}\left\{\prod_1^s \mathrm{F}_q(m_i;\ 1^4)+6\prod_1^s \mathrm{F}_q(m_i;\ 1^2 2)+3\prod_1^s \mathrm{F}_q(m_i;\ 2^2)+8\prod_1^s \mathrm{F}_q(m_i;\ 13)+6\prod_1^s \mathrm{F}_q(m_i;\ 4)\right\}.$$

Section II.

Art. 16. The multipartite partitions which have been under consideration above have involved multipartite parts, and the integers which are constituents of those parts have been quite unrestricted in magnitude. We have now to consider the enumeration when these magnitudes are subject to various restrictions.

The first restriction to come before us is that in which no integer constituent of a multipartite partition is to exceed unity.

We form a fraction $\mathbf{A}_1$ from $\mathbf{Q}_1$ by striking out from the latter every partition which involves a part greater than unity.

Thus

$$\mathbf{A}_1 = 1+(1)+(1^2)+(1^3)+\dots\dots \text{ ad inf.}$$

We now form $\mathbf{A}_2$, $\mathbf{A}_3$, ... $\mathbf{A}_i$, ..., from $\mathbf{A}_1$, by doubling, trebling, ... multiplying by i, ... all the bracket numbers of $\mathbf{A}_1$, in the same way as we formed $\mathbf{Q}_2$, $\mathbf{Q}_3$, ... $\mathbf{Q}_i$, ... from $\mathbf{Q}_1$.

Thus

$$\mathbf{A}_2 = 1+(2)+(2^2)+(2^3)+ \ldots \text{ ad inf.}$$
$$\mathbf{A}_3 = 1+(3)+(3^2)+(3^3)+ \ldots \quad ,, \quad ,,$$
$$\ldots\ldots\ldots \quad ,, \quad ,,$$
$$\mathbf{A}_i = 1+(i)+(i^2)+(i^3)+ \ldots \quad ,, \quad ,,$$
$$\ldots\ldots\ldots \quad ,, \quad ,,$$

We proceed in this manner because we desire the development of the generating symmetric function

$$\frac{1}{(1-a)(1-\alpha a)(1-\beta a)\ldots(1-\alpha\beta a)(1-\alpha\gamma a)\ldots(1-\alpha\beta\gamma a)\ldots(1-\alpha\beta\gamma\delta a)\ldots}$$

there being a denominator factor for every $\alpha, \beta, \gamma, \ldots$ product in which no letter is repeated. The expansion of this fraction involves the whole of the homogeneous product-sums of such $\alpha, \beta, \gamma, \ldots$ products; and we form these product-sums through the medium of the sums of the powers of the products which are, in fact, $\mathbf{A}_1, \mathbf{A}_2, \ldots \mathbf{A}_i, \ldots$. The development is

$$\begin{aligned}
&1+a\mathbf{A}_1\\
&\quad+\frac{a^2}{2\,!}(\mathbf{A}_1{}^2+\mathbf{A}_2)\\
&\quad+\frac{a^3}{3\,!}(\mathbf{A}_1{}^3+3\mathbf{A}_1\mathbf{A}_2+2\mathbf{A}_3)\\
&\quad+\ldots\ldots\\
&\quad+a^k\,\mathrm{F}_k(\mathbf{A})\\
&\quad+\ldots
\end{aligned}$$

where

$$\mathrm{F}_k(\mathbf{A}) = \Sigma\frac{\mathbf{A}_1{}^{k_1}\mathbf{A}_2{}^{k_2}\ldots\mathbf{A}_i{}^{k_i}}{1^{k_1}2^{k_2}\ldots i^{k_i}\,.\,k_1\,!\,k_2\,!\,k_i\,!},$$

precisely similar to the $\mathbf{Q}$ development with $\mathbf{A}$ written for $\mathbf{Q}$.

It may at this point be worth stating that the two developments may be written

$$\exp\left(a\mathbf{Q}_1+\tfrac{1}{2}a^2\mathbf{Q}_2+\tfrac{1}{3}a^3\mathbf{Q}_3+\ldots\right), \qquad \exp\left(a\mathbf{A}_1+\tfrac{1}{2}a^2\mathbf{A}_2+\tfrac{1}{3}a^3\mathbf{A}_3+\ldots\right)$$

respectively.

We now examine the effect of the Hammond operators upon this infinite set of $\mathbf{A}$ functions. It is clear from the well-known fundamental property of the operators that

$$\mathrm{D}_i\mathbf{A}_i = \mathbf{A}_i,$$

and

$$\mathrm{D}_m\mathbf{A}_i = 0 \quad \text{when} \quad m \neq i,$$

results of great simplicity.

When D_m operates upon any product

$$\mathbf{A}_1^{k_1}\mathbf{A}_2^{k_2}\ldots\mathbf{A}_i^{k_i}$$

the demonstration proceeds as with the **Q** function *ante.*

Writing out the **A** product at length and underneath it any composition of m into $k_1+k_2+\ldots+k_i$ parts, zero counting as a part, we note that if the composition operator is to have the effect of multiplying the product by unity and not by zero, every part under the first k_1 factors of the operand must be zero or unity; every part under the next k_2 factors must be zero or 2; every part under the next k_3 factors must be zero or 3; and so on, until finally every part under the last k_i factors must be zero or i.

The number of compositions of m which possess these properties is equal to the coefficients of x^m in the developments of

$$(1+x)^{k_1}(1+x^2)^{k_2}(1+x^3)^{k_3}\ldots(1+x^i)^{k_i},$$

which may be written

$$\left(\frac{1-x^2}{1-x}\right)^{k_1}\left(\frac{1-x^4}{1-x^2}\right)^{k_2}\left(\frac{1-x^6}{1-x^3}\right)^{k_3}\ldots\left(\frac{1-x^{2i}}{1-x^i}\right)^{k_i},$$

or, in CAYLEY's notation,

$$\frac{(2)^{k_1}(4)^{k_2}(6)^{k_3}\ldots(2i)^{k_i}}{(1)^{k_1}(2)^{k_2}(3)^{k_3}\ldots(i)^{k_i}}.$$

Let this coefficient be denoted by

$$F_a(m;\, 1^{k_1}2^{k_2}\ldots i^{k_i})$$

so that

$$D_m\mathbf{A}_1^{k_1}\mathbf{A}_2^{k_2}\ldots\mathbf{A}_i^{k_i} = F_a(m;\, 1^{k_1}2^{k_2}\ldots i^{k_i})\,\mathbf{A}_1^{k_1}\mathbf{A}_2^{k_2}\ldots\mathbf{A}_i^{k_i}.$$

Looking to the symmetric function expressions of $\mathbf{A}_1$, $\mathbf{A}_2\ldots\mathbf{A}_i\ldots$, it will be noted that the only portion of the product

$$\mathbf{A}_1^{k_1}\mathbf{A}_2^{k_2}\ldots\mathbf{A}_i^{k_i},$$

that is free from the elements α, β, $\gamma\ldots$, is unity.

Hence the portion of

$$D_m\mathbf{A}_1^{k_1}\mathbf{A}_2^{k_2}\ldots\mathbf{A}_i^{k_i},$$

that is free from the elements, is

$$F_a(m;\, 1^{k_1}2^{k_2}\ldots i^{k_i});$$

and we may write, as before,

$$(D_m\mathbf{A}_1^{k_1}\mathbf{A}_2^{k_2}\ldots\mathbf{A}_i^{k_i})_{\mathbf{A}=1} = F_a(m;\, 1^{k_1}2^{k_2}\ldots i^{k_i}).$$

The number of partitions of the unipartite number m into k or fewer parts, restricted not to exceed unity, is therefore

$$D_m F_k(\mathbf{A})_{\mathbf{A}=1} = \Sigma \frac{(D_m \mathbf{A}_1^{k_1}\mathbf{A}_2^{k_2}\ldots\mathbf{A}_i^{k_i})_{\mathbf{A}=1}}{1^{k_1}2^{k_2}\ldots i^{k_i}\,.\,k_1!\,k_2!\ldots k_i!},$$

$$= \Sigma \frac{F_a(m\,;\,1^{k_1}2^{k_2}\ldots i^{k_i})}{1^{k_1}2^{k_2}\ldots i^{k_i}\,.\,k_1!\,k_2!\ldots k_i!},$$

$$= \text{coefficients of } x^m \text{ in}$$

$$\Sigma \frac{\left(\frac{1-x^2}{1-x}\right)^{k_1}\left(\frac{1-x^4}{1-x^2}\right)^{k_2}\ldots\left(\frac{1-x^{2i}}{1-x^i}\right)^{k_i}}{1^{k_1}2^{k_2}\ldots i^{k_i}\,.\,k_1!\,k_2!\ldots k_i!},$$

the summation being for every partition

$$k_1+2k_2+\ldots+ik_i$$

of the number k.

Now, obviously, the number we seek is also the coefficient of x^m in $1+x+x^2+\ldots+x^k$. Hence the formula

$$\Sigma \frac{\left(\frac{1-x^2}{1-x}\right)^{k_1}\left(\frac{1-x^4}{1-x^2}\right)^{k_2}\ldots\left(\frac{1-x^{2i}}{1-x^i}\right)^{k_i}}{1^{k_1}2^{k_2}\ldots i^{k_i}\,.\,k_1!\,k_2!\ldots k_i!} = \frac{1-x^{k+1}}{1-x},$$

when k is 3 the identity is

$$\tfrac{1}{6}\{(1+x)^3+3(1+x)(1+x^2)+2(1+x^3)\} = 1+x+x^2+x^3.$$

We have now the result

$$D_{m_1}D_{m_2}\ldots D_{m_s}\,.\,\mathbf{A}_1^{k_1}\mathbf{A}_2^{k_2}\ldots\mathbf{A}_i^{k_i}$$
$$= F_a(m_1\,;\,1^{k_1}2^{k_2}\ldots i^{k_i})\,.\,F_a(m_2\,;\,1^{k_1}2^{k_2}\ldots i^{k_i})\ldots F(m_s\,;\,1^{k_1}2^{k_2}\ldots i^{k_i})\,.\,\mathbf{A}_1^{k_1}\mathbf{A}_2^{k_2}\ldots\mathbf{A}_i^{k_i};$$

and the number of partitions of the multipartite number

$$m_1 m_2 \ldots m_s$$

into k or fewer parts, no integer constituent of the multipartite parts exceeding unity, is

$$\Sigma \frac{F_a(m_1\,;\,1^{k_1}2^{k_2}\ldots i^{k_i})\,.\,F_a(m_2\,;\,1^{k_1}2^{k_2}\ldots i^{k_i})\ldots F_a(m_s\,;\,1^{k_1}2^{k_2}\ldots i^{k_i})}{1^{k_1}2^{k_2}\ldots i^{k_i}.\,k_1!\,k_2!\ldots k_i!};$$

the general solution of the problem.

Some examples are now given.

For the partitions into two, or fewer parts, it is only necessary to consider the cases $m=1$ and $m=2$, since there are no partitions of the nature examined when $m>2$.

$$D_1^{s_1}\tfrac{1}{2}(\mathbf{A}_1^2+\mathbf{A}_2) = \tfrac{1}{2}\,.\,2^{s_1}\mathbf{A}_1^2,$$
$$D_2^{s_2}\tfrac{1}{2}(\mathbf{A}_1^2+\mathbf{A}_2) = \tfrac{1}{2}(\mathbf{A}_1^2+\mathbf{A}_2),$$

so that

$$D_1 \tfrac{1}{2}(\mathbf{A}_1^2+\mathbf{A}_2) = \tfrac{1}{2}\,.\,2\mathbf{A}_1^2,$$

$$D_2 \tfrac{1}{2}(\mathbf{A}_1^2+\mathbf{A}_2) = \tfrac{1}{2}(\mathbf{A}_1^2+\mathbf{A}_2),$$

and

$$D_1^{s_1}D_2^{s_2}\tfrac{1}{2}(\mathbf{A}_1^2+\mathbf{A}_2) = \tfrac{1}{2}\,.\,2^{s_1}\mathbf{A}_1^2,$$

results which show

(i.) that the multipartite number

111 … s_1 times repeated,

has 2^{s_1-1} partitions into two or fewer parts;

(ii.) that the number

222 … s_2 times repeated,

has one partition into two or fewer parts;

(iii.) that the multipartite number

222 … s_2 times, 111 … s_1 times,

has 2^{s_1-1} partitions into two or fewer parts.

Ex. gr. the multipartite number 2111 has the four partitions

(1111 1000), (1011 1100), (1101 1010), (1110 1001).

For the partitions, into three or fewer parts, we have

$$D_3 \tfrac{1}{6}(\mathbf{A}_1^3+3\mathbf{A}_1\mathbf{A}_2+2\mathbf{A}_3) = \tfrac{1}{6}(\mathbf{A}_1^3+3\mathbf{A}_1\mathbf{A}_2+2\mathbf{A}_3),$$

$$D_2 \tfrac{1}{6}(\mathbf{A}_1^3+3\mathbf{A}_1\mathbf{A}_2+2\mathbf{A}_3) = \tfrac{1}{6}(3\mathbf{A}_1^3+3\mathbf{A}_1\mathbf{A}_2),$$

$$D_1 \tfrac{1}{6}(\mathbf{A}_1^3+3\mathbf{A}_1\mathbf{A}_2+2\mathbf{A}_3) = \tfrac{1}{6}(3\mathbf{A}_1^3+3\mathbf{A}_1\mathbf{A}_2),$$

to which we may add for symmetry

$$D_0 \tfrac{1}{6}(\mathbf{A}_1^3+3\mathbf{A}_1\mathbf{A}_2+2\mathbf{A}_3) = \tfrac{1}{6}(\mathbf{A}_1^3+3\mathbf{A}_1\mathbf{A}_2+2\mathbf{A}_3);$$

we gather that

$$D_3^{s_3} \tfrac{1}{6}(\mathbf{A}_1^3+3\mathbf{A}_1\mathbf{A}_2+2\mathbf{A}_3) = \tfrac{1}{6}(\mathbf{A}_1^3+3\mathbf{A}_1\mathbf{A}_2+2\mathbf{A}_3),$$

$$D_2^{s_2} \tfrac{1}{6}(\mathbf{A}_1^3+3\mathbf{A}_1\mathbf{A}_2+2\mathbf{A}_3) = \tfrac{1}{6}(3^{s_2}\mathbf{A}_1^3+3\mathbf{A}_1\mathbf{A}_2),$$

$$D_1^{s_1} \tfrac{1}{6}(\mathbf{A}_1^3+2\mathbf{A}_1\mathbf{A}_2+2\mathbf{A}_3) = \tfrac{1}{6}(3^{s_1}\mathbf{A}_1^3+3\mathbf{A}_1\mathbf{A}_2),$$

$$D_3^{s_3}D_2^{s_2}D_1^{s_1} \tfrac{1}{6}(\mathbf{A}_1^3+3\mathbf{A}_1\mathbf{A}_2+2\mathbf{A}_3) = \tfrac{1}{6}(3^{s_1+s_2}\mathbf{A}_1^3+3\mathbf{A}_1\mathbf{A}_2);$$

and it follows that the multipartite number

333 … s_3 times, 222 … s_2 times, 111 … s_1 times,

has $\tfrac{1}{6}(3^{s_1+s_2}+3)$ partitions, into three or fewer parts, of the nature we are considering.

Ex. gr. The multipartite number 3221 has the five partitions

$$(1111 \quad 1110 \quad 1000), \qquad (1111 \quad 1100 \quad 1010),$$
$$(1110 \quad 1110 \quad 1001), \qquad (1110 \quad 1011 \quad 1100), \qquad (1110 \quad 1101 \quad 1010).$$

Art. 17. Again, to pass to a different restriction, if no integer constituent of a multipartite part is to exceed 2, we strike out from the **Q** functions all partitions which involve integers greater than 2 and arrive at an infinite set of **B** functions which can be dealt with in a similar way. Thus

$$\mathbf{B}_1 = 1+(1)+(2)+(1^2)+(21)+(1^3)+(2^2)+(21^2)+(1^4)+\ldots \quad \text{ad inf.}$$
$$\mathbf{B}_2 = 1+(2)+(4)+(2^2)+(42)+(2^3)+(4^2)+(42^2)+(2^4)+\ldots \quad ,, \quad ,,$$
$$\mathbf{B}_i = 1+(i)+(2i)+(i^2)+(2i,\, i)+(i^3)+(2i,\, 2i)+(2i,\, i,\, i)+(i^4)+\ldots \quad ,, \quad ,,$$

In regard to the Hammond operators

$$D_i\mathbf{B}_i = D_{2i}\mathbf{B}_i = \mathbf{B}_i\,;$$

while every other operator causes $\mathbf{B}_i$ to vanish.

To find the effect of D_m upon the product

$$\mathbf{B}_1^{k_1}\mathbf{B}_2^{k_2}\ldots\mathbf{B}_i^{k_i}$$

we observe that D_m operates through the compositions of m into exactly $k_1+k_2+\ldots+k_i$ parts, zero counting as a part. In order that a particular composition operator shall not cause the product to vanish, the k_s factors of $\mathbf{B}_s^{k_s}$ must only be operated upon by $D_0\,(\equiv 1)$, D_s and D_{2s}. Hence the number of compositions which multiply the product by unity and not by zero is given by the coefficient of x^m in the development of

$$(1+x+x^2)^{k_1}\,(1+x^2+x^4)^{k_2}\ldots(1+x^i+x^{2i})^{k_i},$$

which is

$$\left(\frac{1-x^3}{1-x}\right)^{k_1}\left(\frac{1-x^6}{1-x^2}\right)^{k_2}\ldots\left(\frac{1-x^{3i}}{1-x^i}\right)^{k_i}.$$

This establishes that the effect of D_m upon the product is to multiply it by this coefficient.

The generating function is

$$\begin{aligned} &1+a\mathbf{B}_1 \\ &\quad+\frac{a^2}{2\,!}(\mathbf{B}_1^{\,2}+\mathbf{B}_2) \\ &\quad+\frac{a^3}{3\,!}(\mathbf{B}_1^{\,3}+3\mathbf{B}_1\mathbf{B}_2+2\mathbf{B}_3) \\ &\quad+\ldots \\ &\quad+a^kF_k(\mathbf{B}) \\ &\quad+\ldots \end{aligned}$$

where

$$F_k(\mathbf{B}) = \Sigma \frac{\mathbf{B}_1^{k_1}\mathbf{B}_2^{k_2}\dots\mathbf{B}_i^{k_i}}{1^{k_1}2^{k_2}\dots i^{k_i}\,.\,k_1\,!\,k_2\,!\dots k_i\,!}.$$

Thence we find

$$(D_{m_1}D_{m_2}\dots D_{m_s}F_k(\mathbf{B}))_{B=1}$$

$$= \Sigma \frac{F_b(m_1;\ 1^{k_1}2^{k_2}\dots i^{k_i})\,.\,F_b(m_2;\ 1^{k_1}2^{k_2}\dots i^{k_i})\dots F(m_s;\ 1^{k_1}2^{k_2}\dots i^{k_i})}{1^{k_1}2^{k_2}\dots i^{k_i}\,.\,k_1\,!\,k_2\,!\dots k_i\,!};$$

the solution of the problem of enumeration in respect of the multipartite number $m_1m_2\dots m_s$.

If, in the function $F_k(\mathbf{B})$, we substitute

$$\frac{1-x^{3s}}{1-x^s} \text{ for } \mathbf{B}_s,$$

we obtain

$$\frac{(1-x^{k+1})(1-x^{k+2})}{(1-x)(1-x^2)},$$

because this function enumerates unipartite partitions whose parts are limited in number by k and in magnitude by 2.

As an example consider partitions into three parts. We have the symmetrical results

$$D_6\tfrac{1}{6}(\mathbf{B}_1^3+3\mathbf{B}_1\mathbf{B}_2+2\mathbf{B}_3) = \tfrac{1}{6}(\mathbf{B}_1^3+3\mathbf{B}_1\mathbf{B}_2+2\mathbf{B}_3),$$

$$D_5\tfrac{1}{6}(\mathbf{B}_1^3+3\mathbf{B}_1\mathbf{B}_2+2\mathbf{B}_3) = \tfrac{1}{6}(3\mathbf{B}_1^3+3\mathbf{B}_1\mathbf{B}_2),$$

$$D_4\tfrac{1}{6}(\mathbf{B}_1^3+3\mathbf{B}_1\mathbf{B}_2+2\mathbf{B}_3) = \tfrac{1}{6}(6\mathbf{B}_1^3+3\,.\,2\mathbf{B}_1\mathbf{B}_2),$$

$$D_3\tfrac{1}{6}(\mathbf{B}_1^3+3\mathbf{B}_1\mathbf{B}_2+2\mathbf{B}_3) = \tfrac{1}{6}(7\mathbf{B}_1^3+3\mathbf{B}_1\mathbf{B}_2+2\mathbf{B}_3),$$

and D_2, D_1, D_0, yield the same results as D_4, D_5, D_6, respectively.

The number m, not exceeding 6, D_m and D_{6-m} produce upon the operand the same result. This symmetry naturally follows from the known property of the function

$$\frac{(1-x^{k+1})(1-x^{k+2})}{(1-x)(1-x^2)}$$

which on expansion is, as regards coefficients, centrally symmetrical.

We now at once deduce that

$$D_6^sF_3(\mathbf{B}) = D_0^sF_3(\mathbf{B}) = \tfrac{1}{6}(\mathbf{B}_1^3+3\mathbf{B}_1\mathbf{B}_2+2\mathbf{B}_3)$$

$$D_5^sF_3(\mathbf{B}) = D_1^sF_3(\mathbf{B}) = \tfrac{1}{6}(3^s\mathbf{B}_1^3+3\mathbf{B}_1\mathbf{B}_2)$$

$$D_4^sF_3(\mathbf{B}) = D_2^sF_3(\mathbf{B}) = \tfrac{1}{6}(6^s\mathbf{B}_1^3+3\,.\,2^s\mathbf{B}_1\mathbf{B}_2)$$

$$D_3^sF_3(\mathbf{B}) = \tfrac{1}{6}(7^s\mathbf{B}_1^s+3\mathbf{B}_1\mathbf{B}_2+2\mathbf{B}_3).$$

We deduce that the multipartite number

$$666 \dots s \text{ times}$$

has only one partition of the nature we consider, and this of course is quite obvious.

That the multipartite numbers

$$555 \dots s \text{ times}, \qquad 111 \dots s \text{ times}$$

have each of them

$$\tfrac{1}{6}(3^s+3) \text{ partitions};$$

the multipartite numbers

$$444 \dots s \text{ times}, \qquad 222 \dots s \text{ times},$$

have each of them

$$\tfrac{1}{6}(6^s+3 \,.\, 2^s) \text{ partitions};$$

and the number

$$333 \dots s \text{ times}$$

$$\tfrac{1}{6}(7^s+5) \text{ partitions.}$$

Also the multipartite number

$$333 \dots s \text{ times}, \; 222 \dots t \text{ times}$$

has

$$\tfrac{1}{6}(7^s \,.\, 6^t+3 \,.\, 2^s)$$

partitions, and various other results.

The symmetry shown above in the case of multipartite numbers is of general application in the subject and is very remarkable. I do not see any other *a priori* proof of it at the moment of writing.

Art. 18. In general, we consider the case in which no constituent of the multipartite parts is to exceed the integer j. We strike out from the functions $\mathbf{Q}$ all partitions which involve numbers exceeding j and reach the infinite series of functions

$$\mathbf{J}_1, \mathbf{J}_2, \dots \mathbf{J}_i \dots .$$

These functions are operated upon in the manner

$$\mathbf{D}_i\mathbf{J}_i = \mathbf{D}_{2i}\mathbf{J}_i = \mathbf{D}_{3i}\mathbf{J}_i = \dots = \mathbf{D}_{ji}\mathbf{J}_i = \mathbf{J}_i;$$

while every other Hammond operator causes $\mathbf{J}_i$ to vanish. By the same reasoning as was used in the special cases we find that

$$\mathbf{D}_m\mathbf{J}_1^{k_1}\mathbf{J}_2^{k_2} \dots \mathbf{J}_i^{k_i} = \mathrm{F}_j(m;\; 1^{k_1}2^{k_2} \dots i^{k_i}) \,.\, \mathbf{J}_1^{k_1}\mathbf{J}_2^{k_2} \dots \mathbf{J}_i^{k_i},$$

where

$$\mathrm{F}_j(m;\; 1^{k_1}2^{k_2} \dots i^{k_i})$$

is equal to the coefficient of x^m in the expansion of the function

$$(1+x+\dots+x^j)^{k_1}(1+x^2+\dots+x^{2j})^{k_2} \dots (1+x^i+\dots+x^{ij})^{k_i},$$

which, written in CAYLEY's notation, is

$$\left\{\frac{(j+1)}{(1)}\right\}^{k_1}\left\{\frac{(2j+2)}{(2)}\right\}^{k_2}\ldots\left\{\frac{(ij+i)}{(i)}\right\}^{k_i}.$$

We have

$$\mathrm{F}_k(\mathbf{J}) = \Sigma\frac{\mathbf{J}_1^{k_1}\mathbf{J}_2^{k_2}\ldots\mathbf{J}_i^{k_i}}{1^{k_1}2^{k_2}\ldots i^{k_i}.\,k_1!\,k_2!\ldots k_i!};$$

and if, in $\mathrm{F}_k(\mathbf{J})$, we write

$$\frac{(sj+s)}{(s)} \text{ for } \mathbf{J}_s$$

we must reach the expression

$$\frac{(k+1)(k+2)\ldots(k+j)}{(1)(2)\ldots(j)} \text{ otherwise written } \frac{(j+1)(j+2)\ldots(j+k)}{(1)(2)\ldots(k)}.$$

For the rest the enumeration, connected with this restriction, proceeds *pari passu* with the special cases involving the values of j, ∞, 1, 2, which have been already examined.

Art. 19. Coming now to the last stage of the investigation we have for consideration the partitions of multipartite numbers in which the constituents of the multipartite parts *must* be drawn from the particular set of numbers

$$u_1,\ u_2,\ \ldots\ u_s.$$

The partitions involved in the first function $\mathbf{U}_1$ of the infinite series

$$\mathbf{U}_1,\ \mathbf{U}_2,\ \mathbf{U}_3,\ \ldots\ \mathbf{U}_k,\ \ldots$$

must involve the numbers u_1, u_2, … u_s, and no others.

Thus

$$\mathbf{U}_1 = 1+(u_1)+(u_1^2)+(u_2)+(u_1^3)+(u_1u_2)+(u_3)+\ldots \text{ ad inf.},$$

$$\mathbf{U}_2 = 1+(2u_1)+(2u_12u_1)+(2u_2)+(2u_12u_12u_1)+(2u_12u_2)+(2u_3)+\ldots \text{ ad inf.},$$

. . . .

$$\mathbf{U}_i = 1+(iu_1)+(iu_1iu_1)+(iu_2)+(iu_1iu_1iu_1)+(iu_1iu_2)+(iu_3)+\ldots \text{ ad inf.}$$

Clearly

$$\mathrm{D}_{u_1}\mathbf{U}_1 = \mathrm{D}_{u_2}\mathbf{U}_1 = \ldots = \mathrm{D}_{u_s}\mathbf{U}_1 = \mathbf{U}_1;$$

and any other Hammond operator causes $\mathbf{U}_1$ to vanish.

Also

$$\mathrm{D}_{iu_1}\mathbf{U}_i = \mathrm{D}_{iu_2}\mathbf{U}_i = \ldots = \mathrm{D}_{iu_s}\mathbf{U}_i = \mathbf{U}_i;$$

and any other Hammond operator causes $\mathbf{U}_i$ to vanish.

We have presented to us the generating function

$$\begin{aligned}&1+a\mathbf{U}_1\\&+\frac{a^2}{2!}(\mathbf{U}_1^{\,2}+\mathbf{U}_2)\\&+\frac{a^3}{3!}(\mathbf{U}_1^{\,3}+3\mathbf{U}_1\mathbf{U}_2+2\mathbf{U}_3)\\&+\ldots\\&+a^k\mathrm{F}^k(\mathbf{U})\\&+\ldots\end{aligned}$$

where

$$\mathrm{F}_k(\mathbf{U})=\Sigma\frac{\mathbf{U}_1^{\,k_1}\mathbf{U}_2^{\,k_2}\ldots\mathbf{U}_i^{\,k_i}}{1^{k_1}.\,2^{k_2}\ldots i^{k_i}.\,k_1!\,k_2!\ldots k_i!}.$$

The effect of D_m upon the product

$$\mathbf{U}_1^{\,k_1}\mathbf{U}_2^{\,k_2}\ldots\mathbf{U}_i^{\,k_i}$$

is to multiply it by the coefficient of x^m in the function

$$(1+x^{u_1}+x^{u_2}+\ldots+x^{u_s})^{k_1}(1+x^{2u_1}+x^{2u_2}+\ldots+x^{2u_s})^{k_2}\ldots(1+x^{iu_1}+x^{iu_2}+\ldots+x^{iu_s})^{k_i},$$

which I write in the abbreviated notation

$$\mathrm{X}_u^{\,k_1}\mathrm{X}_{2u}^{\,k_2}\ldots\mathrm{X}_{iu}^{\,k_i}.$$

If in the expression of $\mathrm{F}_k(\mathbf{U})$ we write X_{su} for $\mathbf{U}_s$ we must reach the expression which enumerates the partitions of unipartite numbers into k or fewer parts, such parts being drawn from the series $u_1, u_2, u_3\ldots u_s$. That is to say we must arrive at the coefficient of a^k in the expansion of

$$\frac{1}{(1-a)(1-ax^{u_1})(1-ax^{u_2})\ldots(1-ax^{u_s})}.$$

Hence this coefficient has the expression

$$\Sigma\frac{\mathrm{X}_u^{\,k_1}\mathrm{X}_{2u}^{\,k_2}\ldots\mathrm{X}_{iu}^{\,k_i}}{1^{k_1}.\,2^{k_2}\ldots i^{k_i}.\,k_1!\,k_2!\ldots k_i!}.$$

To enumerate the partitions we have

$$\mathrm{D}_{m_1}\mathrm{F}_k(\mathbf{U})=\Sigma\frac{\mathrm{F}_u(m_1;\,1^{k_1}2^{k_2}\ldots i^{k_i})}{1^{k_1}.\,2^{k_2}\ldots i^{k_i}.\,k_1!\,k_2!\ldots k_i!}\mathbf{U}_1^{\,k_1}\mathbf{U}_2^{\,k_2}\ldots\mathbf{U}_i^{\,k_i},$$

where $F_u(m_1;\ 1^{k_1}2^{k_2}\dots i^{k_i})$ is equal to the coefficient of x^{m_1} in

$$X_u^{k_1}X_{2u}^{k_2}\dots X_{iu}^{k_i}.$$

$$D_{m_1}D_{m_2}\dots D_{m_s}F_k(\mathbf{U})$$
$$= \Sigma \frac{F_u(m_1;\ 1^{k_1}2^{k_2}\dots i^{k_i}) \,.\, F_u(m_2;\ 1^{k_1}2^{k_2}\dots i^{k_i}) \dots F(m_s;\ 1^{k_1}2^{k_2}\dots i^{k_i})}{1^{k_1}\,.\,2^{k_2}\dots i^{k_i}\,.\,k_1\,!\,k_2\,!\dots k_i\,!} \,.\, \mathbf{U}_1^{k_1}\mathbf{U}_2^{k_2}\dots \mathbf{U}_i^{k_i}\,;$$

and thence the number of partitions of the multipartite number

$$m_1m_2\dots m_s,$$

into k or fewer parts, such parts being drawn exclusively from the series

$$u_1,\ u_2,\ \dots u_s,$$

is

$$\Sigma \frac{F_u(m_1;\ 1^{k_1}2^{k_2}\dots i^{k_i}) \,.\, F_u(m_2;\ 1^{k_1}2^{k_2}\dots i^{k_i}) \dots F_u(m_s;\ 1^{k_1}2^{k_2}\dots i^{k_i})}{1^{k_1}\,.\,2^{k_2}\dots i^{k_i}\,.\,k_1\,!\,k_2\,!\dots k_i\,!}$$

the summation being in regard to the partitions of k.

As an example, I will consider partitions of multipartite numbers where the numbers which are constituents of the multipartite parts are limited to be either 3, 5, or 7. For the partitions into three or fewer parts we have the function

$$\tfrac{1}{6}(\mathbf{U}_1^3+3\mathbf{U}_1\mathbf{U}_2+2\mathbf{U}_3),$$

and we have to find the coefficients of x^m in the three functions

$$(1+x^3+x^5+x^7)^3,$$
$$(1+x^3+x^5+x^7)(1+x^6+x^{10}+x^{14}),$$
$$(1+x^9+x^{15}+x^{21}).$$

Thence, as particular cases,

$$\mathbf{D}_{12}\mathbf{U}_1^3 = 6\mathbf{U}_1^3;\quad \mathbf{D}_{11}\mathbf{U}_1^3 = 3\mathbf{U}_1^3;\quad \mathbf{D}_{10}\mathbf{U}_1^3 = 9\mathbf{U}_1^3;$$
$$\mathbf{D}_{12}\mathbf{U}_1\mathbf{U}_2 = 0;\quad \mathbf{D}_{11}\mathbf{U}_1\mathbf{U}_2 = \mathbf{U}_1\mathbf{U}_2;\quad \mathbf{D}_{10}\mathbf{U}_1\mathbf{U}_2 = 0;$$
$$\mathbf{D}_{12}\mathbf{U}_3 = \mathbf{D}_{11}\mathbf{U}_3 = \mathbf{D}_{10}\mathbf{U}_3 = 0.$$

Thence

$$\mathbf{D}_{12}^{\sigma_1}\tfrac{1}{6}(\mathbf{U}_1^3+3\mathbf{U}_1\mathbf{U}_2+2\mathbf{U}_3) = \tfrac{1}{6}\,.\,6^{\sigma_1}\mathbf{U}_1^3,$$
$$\mathbf{D}_{11}^{\sigma_2}\tfrac{1}{6}(\mathbf{U}_1^3+3\mathbf{U}_1\mathbf{U}_2+2\mathbf{U}_3) = \tfrac{1}{6}(6^{\sigma_2}\mathbf{U}_1^3+3\mathbf{U}_1\mathbf{U}_2),$$
$$\mathbf{D}_{10}^{\sigma_3}\tfrac{1}{6}(\mathbf{U}_1^3+3\mathbf{U}_1\mathbf{U}_2+2\mathbf{U}_3) = \tfrac{1}{6}\,.\,9^{\sigma_3}\mathbf{U}_1^3,$$

showing that the multipartite numbers

$$12^{\sigma_1},\qquad 11^{\sigma_2},\qquad 10^{\sigma_3},$$

have 6^{σ_1-1}, $\frac{1}{6}(3^{\sigma_2}+3)$, $\frac{1}{6}\,.\,9^{\sigma_3}$ partitions respectively into three or fewer parts.

Also

$$\mathbf{D}_{12}^{\sigma_1}\mathbf{D}_{11}^{\sigma_2}\mathbf{D}_{10}^{\sigma_3}\tfrac{1}{6}(\mathbf{U}_1^3+3\mathbf{U}_1\mathbf{U}_2+2\mathbf{U}_3) = \tfrac{1}{6}(6^{\sigma_1}\,.\,3^{\sigma_2}\,.\,9^{\sigma_3}\mathbf{U}_1^3+3\mathbf{U}_1\mathbf{U}_2),$$

where the term $3\mathbf{U}_1\mathbf{U}_2$ only appears if σ_1 and σ_2 are *both* zero.

Hence in general, if σ_1 and σ_2 are not both zero, the multipartite number

$$12^{\sigma_1}11^{\sigma_2}10^{\sigma_3}$$

possesses $6^{\sigma_1-1}\,.\,3^{\sigma_2}\,.\,9^{\sigma_3}$ partitions of the nature considered. In particular the multipartite number

$$12\ \ 12\ \ 11,\quad \text{for } \sigma_1 = 2,\ \sigma_2 = 1,\ \sigma_3 = 0,$$

possesses the 18 partitions

(775 553 003),	(755 573 003),	(573 705 053),
(775 053 503),	(575 753 003),	(753 073 505),
(773 555 003),	(755 073 503),	(573 703 055),
(773 553 005),	(575 703 053),	(555 073 703),
(773 505 053),	(753 573 005),	(553 705 073),
(773 055 503),	(753 075 503),	(553 075 703).

Art. 20. In considering the partitions of the multipartite number $m_1m_2\ldots m_s$ the partitions of the unipartite constituents of the number have been regarded as being subject to the same conditions and restrictions. This, however, is not necessary except in the case of the number of parts which has been denoted by k.

We may for m_1 choose any of the restrictions that have been denoted by the symbols **A**, **B**, ... **J** ... **Q**, **U**. For m_2 similarly and so on. For instance, suppose the numbers m_1, m_2, m_3 are subject to the restrictions denoted by **B**, **Q**, and **U**; that is to say, the partitions of m_1 are such that no part exceeds 2; the partitions of m_2 are unrestricted; the partitions of m_3 are such that the parts are drawn exclusively from specified integers $u_1, u_2, \ldots u_s$.

For the partitions of the multipartite number $m_1m_2\ldots m_s$, subject to this combination of restrictions, we

(i.) Take

$$\mathrm{D}_{m_1}\sum\frac{\mathbf{B}_1^{k_1}\mathbf{B}_2^{k_2}\ldots\mathbf{B}_i^{k_i}}{1^{k_1}2^{k_2}\ldots i^{k_i}\,.\,k_1!\,k_2!\ldots k_i!},$$

with the result

$$\sum\mathrm{F}_b(m_1;\ 1^{k_1}2^{k_2}\ldots i^{k_i})\frac{\mathbf{B}_1^{k_1}\mathbf{B}_2^{k_2}\ldots\mathbf{B}_i^{k_i}}{1^{k_1}2^{k_2}\ldots i^{k_i}\,.\,k_1!\,k_2!\ldots k_i!};$$

(ii.) We change the symbol $\mathbf{B}$ into the symbol $\mathbf{Q}$ and find

$$D_{m_2} \Sigma F_b(m_1;\ 1^{k_1}2^{k_2}\dots i^{k_i}) \frac{\mathbf{Q}_1^{k_1}\mathbf{Q}_2^{k_2}\dots \mathbf{Q}_i^{k_i}}{1^{k_1}2^{k_2}\dots i^{k_i}\,.\,k_1!\,k_2!\dots k_i!}$$

$$= \Sigma F_b(m_1;\ 1^{k_1}2^{k_2}\dots i^{k_i})\,.\,F_q(m_2;\ 1^{k_1}2^{k_2}\dots i^{k_i}) \frac{\mathbf{Q}_1^{k_1}\mathbf{Q}_2^{k_2}\dots \mathbf{Q}_i^{k_i}}{1^{k_1}2^{k_2}\dots i^{k_i}\,.\,k_1!\,k_2!\dots k_i!};$$

(iii.) Lastly we change the symbol $\mathbf{Q}$ into the symbol $\mathbf{U}$ and find that

$$D_{m_3} \Sigma F_b(m_1;\ 1^{k_1}2^{k_2}\dots i^{k_i})\,.\,F_q(m_2;\ 1^{k_1}2^{k_2}\dots i^{k_i}) \frac{\mathbf{U}_1^{k_1}\mathbf{U}_2^{k_2}\dots \mathbf{U}_i^{k_i}}{1^{k_1}2^{k_2}\dots i^{k_i}\,.\,k_1!\,k_2!\dots k_i!}$$

$$= \Sigma F_b(m_1;\ 1^{k_1}2^{k_2}\dots i^{k_i})\,.\,F_q(m_2;\ 1^{k_1}2^{k_2}\dots i^{k_i})\,.\,F_u(m_3;\ 1^{k_1}2^{k_2}\dots i^{k_i}) \frac{\mathbf{U}_1^{k_1}\mathbf{U}_2^{k_2}\dots \mathbf{U}_i^{k_i}}{1^{k_1}2^{k_2}\dots i^{k_i}\,.\,k_1!\,k_2!\dots k_i!},$$

and herein putting $\mathbf{U}_1 = \mathbf{U}_2 = \dots = \mathbf{U}_i = 1$, we reach the conclusion that the multipartite number $m_1m_2\dots m_s$ has

$$\Sigma \frac{F_b(m_1;\ 1^{k_1}2^{k_2}\dots i^{k_i})\,.\,F_q(m_2;\ 1^{k_1}2^{k_2}\dots i^{k_i})\,.\,F_u(m_3;\ 1^{k_1}2^{k_2}\dots i^{k_i})}{1^{k_1}2^{k_2}\dots i^{k_i}\,.\,k_1!\,k_2!\dots k_i!}$$

partitions of the nature we are considering